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Multiplicative cascades applied to PDEs (two numerical examples)

Jorge M. Ramirez *

Department of Mathematics, Oregon State University, Kidder Hall 324, Corvallis, OR 97331-4605, USA

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Abstract

Numerical approximations to the Fourier transformed solution of partial differential equations are obtained via Monte Carlo simulation of certain random multiplicative cascades. Two particular equations are considered: linear diffusion equation and viscous Burgers equation. The algorithms proposed exploit the structure of the branching random walks in which the multiplicative cascades are defined. The results show initial numerical approximations with errors less than 5% in the leading Fourier coefficients of the solution. This approximation is then improved substantially using a Picard iteration scheme on the integral equation associated with the representation of the respective PDE in Fourier space.

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1. Introduction

Stochastic processes have been found to have important connections to deterministic partial differential equations (PDEs), most notable being the relationship between the linear diffusion equation and Brownian motion through the Feynman–Kac formula (e.g. see [1,12]). In later developments, probabilistic representations have been found to solutions of semilinear and quasilinear equations. Important examples include the well-known example by McKean [2] for the solution to the KPP equation as the expected value of a functional of branching Brownian motion, and more recently the work in [3], where the solution to the incompressible Navier–Stokes equations is written in terms of a jumping and branching Brownian motion.

In each of the three works cited above, the solution to the PDE is represented as the expected value of a functional acting on the sample paths of certain stochastic process evolving in physical space. This paper deals with the analogous idea in *Fourier space*, along the lines of the method introduced in [9] for the Navier–Stokes equations. Namely, multiplicative functionals of tree-like stochastic models are used to give probabilistic

* Tel.: +1 541 737 4686.
E-mail address: jramirez@math.oregonstate.edu.

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representation of the Fourier transform of the solution to the PDE. Two particular examples are considered: simple linear diffusion with a potential and viscous Burgers equation. Further restrictions are imposed to the Fourier transform of the data in each PDE to achieve the probabilistic representation.

The main emphasis is on the design of *Monte Carlo simulation* schemes to numerically approximate the solution of each PDE in Fourier space. The initial approximations are further improved by means of numerical *Picard iteration*.

In order to fix the main ideas of the methods used in subsequent sections of this paper, consider the simple example of the diffusion equation with Fickian flux and a sink term of constant rate $c > 0$,

$$u_t = u_{xx} + cu, \quad t > 0, \ u(0^+) = u_0. \tag{1}$$

Feynman–Kac's formula gives the following explicit probabilistic representation of the solution u in physical space,

$$u(t,x) = \int_{-\infty}^{\infty} e^{-ct} \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}(x-y)^2} u_0(y)\, dy = \mathbb{E}_x\{\mathbf{1}_{[S>t]} u_0(B_t)\}, \tag{2}$$

where $B = \{B_t : t \geqslant 0\}$ is a standard Brownian motion and S is an exponentially distributed random variable independent of B, namely $\mathbb{P}(S > t) = e^{-ct}$, $t > 0$. The symbol $\mathbb{E}_x$ denotes expectation conditioned to the event $[B_0 = x]$ and $\mathbf{1}_{[S>t]}$ denotes the indicator function of the event $[S > t]$, i.e., $\mathbf{1}_{[S>t]}$ is 1 or 0 depending on wether $[S > t]$ occurs or not.

The probabilistic representation in (2) helps create a very clear and intuitive physical picture of the solution to the PDE: for $t > 0$, particles initially distributed according to u_0, start moving (diffusing) following the paths of B and are "killed" at a random time S; $u(x,t)$ is the expected fraction of surviving particles that occupy the point x at time t.

A generalization of the example (1) will be revisited in Section 2, but in contrast to the description above, the solution is now represented probabilistically in Fourier space. Namely, the Fourier transform of u

$$\hat{u}(t,\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} u(t,x)\, dx,$$

which from (1) satisfies

$$\hat{u}_t(t,\xi) = -\xi^2 \hat{u}(t,\xi) + c\hat{u}(t,\xi), \quad t > 0, \ \hat{u}(0^+) = \hat{u}_0, \tag{3}$$

is expressed as the expected value of a functional of some process that takes values in time-frequency space.

The problem in (3) can be solved exactly in a variety of ways. One non-obvious method that illustrates the main ideas used in this paper is outlined below.

Multiply both sides of (3) by $e^{\xi^2 t}$ and integrate on $(0, t)$ to get,

$$\hat{u}(t,\xi) = e^{-\xi^2 t}\hat{u}_0(\xi) + \frac{c}{\xi^2} \int_0^t \xi^2 e^{-\xi^2 s} \hat{u}(t-s,\xi)\, ds, \tag{4}$$

$$= \mathbb{E}\left\{ \hat{u}_0(\xi)\mathbf{1}_{[S_0>t]} + \frac{c}{\xi^2} \hat{u}(t - S_0, \xi)\mathbf{1}_{[S_0<t]} \right\}, \tag{5}$$

where S_0 is an exponentially distributed with

$$\mathbb{P}(S_0 > t) = e^{-\xi^2 t}. \tag{6}$$

The goal is now to use (5) to construct a stochastic model τ and a functional $\mathsf{X} = \mathsf{X}(\tau)$ such that

$$\hat{u}(t,\xi) = \mathbb{E}\mathsf{X}(\tau). \tag{7}$$

Consider a root vertex $\langle 0 \rangle$. Assign to it a frequency (or type) ξ, and compare a realization of the random time S_0 to the termination time of $\langle 0 \rangle$ denoted by $T_0 = t$. In the event that $S_0 \geqslant t$, the process stops and the multiplier $M_0 = u_0(\xi)$ is assigned to $\langle 0 \rangle$. If $S_0 < t$, a new vertex $\langle 1 \rangle$ of τ is created. In this example, the frequency of the vertex $\langle 1 \rangle$ remains equal to ξ, i.e., is selected according to the Dirac delta distribution δ_ξ. The respective multiplier is $M_1 = \frac{c}{\xi^2}$. The construction of τ is continued by generating an exponential time S_1 independent of, and with the same distribution (6), as S_0. The termination time of the vertex $\langle 1 \rangle$ is set to $T_1 = t - S_0$.

Let N be the number of vertices in τ, and define X as the following *multiplicative functional*

$$\mathsf{X}(\tau) = \prod_{i=0}^{N-1} M_i. \tag{8}$$

The formulation is then completed by showing that indeed (7) holds. This can be done following the lines of the proof of Theorem 2.1.

The probabilistic formulations given in (5) or (7) do not immediately point to a physical picture as clear as the one obtained by Feynman–Kac's formula (2) in physical space. Nevertheless, the stochastic model (8) gives a useful way of studying solutions in Fourier space and, in particular, is amenable to numerically estimate $\hat{u}$. Two techniques are used in this paper to achieve this, and can be illustrated with the derivations in the example above. The first is direct Monte Carlo simulation of the expected value in (7). The second is more classical and stems from understanding (4) as the fixed point equation $\hat{u}(t,\xi) = F(\hat{u},t,\xi)$, where F is the linear operator given by the right-hand side of (4) on an appropriate Banach space. Provided that F is a contraction in the norm of the space under consideration, a solution is given by $\hat{u} = \lim_{n\to\infty}\hat{u}^{(n)}$ with $\hat{u}^{(n+1)} = F(\hat{u}^{(n)},t,\xi)$, $n = 0,1,2,\ldots$ (see [4] for more details). This latter method is referred to in this paper as Picard iteration.

A probabilistic formulation along the lines of the illustrative example (1) is available for a diverse class of evolution equations, including reaction–diffusion, Schrödinger, Burgers and Navier–Stokes equations (see [5–8]). In the seminal work of [9], the authors show that the Fourier transformed solution to the incompressible Navier–Stokes equations in three dimensions satisfies an equation of the form of (7) with τ having tree graph structure. The branching at the vertices of τ is produced by the quadratic nonlinearity in the PDE, very much like the branching necessary for McKean's solution to KPP equation in physical space (see [2]).

The stochastic process involved in the probabilistic representation of Navier–Stokes in Fourier space is difficult to model due to the three degrees of freedom in the frequencies assigned to the vertices of τ. Fortunately, the main features of the associated multiplicative random functional appear also in the probabilistic formulation of the one-dimensional Burgers equation in Fourier space. The branching at the vertices of τ is linked to the nonlinear term common to both equations. The example of Burgers equation is worked out in detail in Section 3.

In the probabilistic formulation for Burgers and the Navier–Stokes equations, the branching structure requires that one specifies the probability distribution of the frequency of the offspring vertices given the frequency of the parent vertex. In the case of the diffusion equation (1), where τ has no branching, this distribution was a Dirac delta distribution. Admissible distributions are referred to as *majorizing kernels*, were studied thoroughly in [10] for the case of incompressible Navier–Stokes equations. The authors also show how the majorizing kernel can be used in establishing existence and regularity of solutions.

Although the theory and numerical schemes explored here are mathematically motivated, there is hope for a physical intuitive picture for the branching stochastic model in the probability formulation of Burgers or Navier–Stokes equations in Fourier space. The branching process τ, together with the multipliers associated to its vertices, is referred to as a *stochastic cascade* (see [5] and references therein). This name is appropriate, especially when viewed in terms of Kolmogorov's statistical theory of turbulence. Identifying the actual link between the probabilistic formulation and Kolmogorov's cascade is still a very important open problem. It is conceivable that the majorizing kernels that determine the branching in τ are related to the physical rates of transport of energy between frequencies in turbulent flows. The numerical models reported in this paper are simple tools that might help to shed light on this relationship.

The organization is as follows. The rest of this introduction settles down some terminology on the main tools to be used, namely Fourier transform and tree graphs. The linear diffusion equation with potential

$$u_t = \frac{a^2}{2}u_{xx} + c(x)u$$

is considered in Section 2, and Burgers equation

$$u_t + \sqrt{2\pi}uu_x = vu_{xx} + f(t,x)$$

is treated on Section 3.

Some remarks on notation are in order. All partial differential equations are assumed to hold in the Schwartz class of tempered distributions $\mathscr{S}'$. The Fourier transform on $\mathscr{S}'$ is defined through its action on test functions as

$$\hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}x\xi}\phi(x)\,\mathrm{d}x, \quad \xi \in \mathbb{R}, \;\; \phi \in C_0^{\infty}(\mathbb{R}).$$

In the case of periodic distributions, the Fourier series representation is available,

$$u = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \hat{u}(k)\mathrm{e}^{\mathrm{i}kx}, \quad u \in \mathscr{S}', \;\; u \text{ periodic}. \tag{9}$$

Details on the construction of the Fourier transform on $\mathscr{S}'$ and its properties can be found in [4] or [11].

Tree graphs are used in this paper as a suitable frame to define certain multiplicative processes, and only a bit of special notation is required. A tree graph τ is a connected graph of vertices with no cycles, and with vertex set containing a unique root vertex coded as $\langle 0\rangle$. The edges of a tree are determined by the relation "belongs to the offspring of". The $n_{\langle 0\rangle}$ vertices in the offspring of $\langle 0\rangle$ are coded by $\langle 1\rangle, \langle 2\rangle, \ldots, \langle n_{\langle 0\rangle}\rangle$. The vertices of second generation, i.e., the offspring of some $\langle i\rangle$, are coded as $\langle i1\rangle, \langle i2\rangle, \ldots, \langle in_{\langle i\rangle}\rangle$. Inductively, a vertex of the nth generation of τ has the form $\langle v\rangle = \langle i_1 i_2 \ldots i_n\rangle$, where i_k is a positive integer for each k. For a vertex $\langle v\rangle = \langle i_1 i_2 \ldots i_n\rangle$its length is defined as $|\langle v\rangle| = n$ with the convention $|\langle 0\rangle| = 0$. For $j = 1, 2, \ldots, |\langle v\rangle|$, the jth level of $\langle v\rangle$ is the vertex let $\langle v\rangle|j = \langle i_1, \ldots, i_j\rangle$, $\langle v\rangle|0 = \langle 0\rangle$. The sequence $\{\langle v\rangle|0, \langle v\rangle|1, \ldots, \langle v\rangle\}$ can be viewed as a path connecting $\langle 0\rangle$ with $\langle v\rangle$.

2. Linear diffusion equation

Consider the one-dimensional linear diffusion equation in $(-\infty, \infty)$ with constant diffusion coefficient $\frac{1}{2}a^2 > 0$, potential $c = c(x)$ and initial condition $u_0 = u_0(x)$,

$$u_t(t) = \frac{a^2}{2}u_{xx}(t) + cu(t), \quad t > 0 \;\; u(0^+) = u_0. \tag{10}$$

Assume that c and u_0 have Fourier series with a finite number of terms, and that the Fourier coefficients of c are all nonnegative. In particular, c and u_0 are periodic and there exist finite sets of frequencies $\{\alpha_i\}_{i=1}^{m_c}$ and $\{\beta_j\}_{j=1}^{m_u}$, such that

$$c(x) = \frac{1}{\sqrt{2\pi}} \sum_{i=1}^{m_c} \hat{c}(\alpha_i)\mathrm{e}^{\mathrm{i}\alpha_i x}, \quad u_0(x) = \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{m_u} \hat{u}_0(\beta_j)\mathrm{e}^{\mathrm{i}\beta_j x}. \tag{11}$$

Assume furthermore that

$$\hat{c}(\alpha_i) > 0, \quad i = 1, \ldots, m_c, \qquad \hat{u_0}(\beta_j) \neq 0, \quad j = 1, \ldots, m_u.$$

Take Fourier transform on both sides of Eq. (10) and use the integration factor $\mathrm{e}^{\frac{a^2}{2}\xi^2 s}$ to get

$$\hat{u}(t,\xi) = \mathrm{e}^{-\frac{a^2}{2}\xi^2 t}\hat{u}_0(\xi) + \int_0^t \mathrm{e}^{-\frac{a^2}{2}\xi^2 s}\left\{\sum_{i=1}^{m_c} \hat{c}(\alpha_i)\hat{u}(t-s, \xi-\alpha_i)\right\}\mathrm{d}s. \tag{12}$$

The integral equation (12) can readily be written as the expected value of some multiplicative process, however, for computational reasons described in (2.1), is more convenient to introduce an exponential factor whose argument does not depend on ξ, for example e^{-t}. Define

$$m(t,\xi) = \mathrm{e}^{-\frac{a^2}{2}\xi^2 t + t}, \quad m'(t,\xi) = m(t,\xi)\sum_{i=1}^{m_c} \hat{c}(\alpha_i) \tag{13}$$

and write

$$\hat{u}(t,\xi) = m(t,\xi)\mathrm{e}^{-t}\hat{u}_0(\xi) + \int_0^t \mathrm{e}^{-s}m'(s,\xi)\sum_{k=1}^{m_c}\left[\frac{\hat{c}(\alpha_k)}{\sum_{i=1}^{m_c}\hat{c}(\alpha_i)}\hat{u}(t-s, \xi-\alpha_i)\right]\mathrm{d}s, \tag{14}$$

The representation in (14) suggests the following stochastic model for $\hat{u}$. For each $t > 0$, $\xi \in \mathbb{R}$, consider the linear tree graph

$$\tau(t,\xi) = \{\langle 0\rangle, \langle 1\rangle, \langle 11\rangle, \langle 111\rangle, \ldots\} = \{\langle 0\rangle, \langle 1\rangle, \langle 2\rangle, \langle 3\rangle, \ldots\}$$

and let $(\Omega, \mathscr{F}, \mathbb{P}_\xi)$ be a probability space. To each vertex $\langle i\rangle$ associate a Fourier wavenumber (or type) $\xi_{\langle i\rangle}$ and a exponential random time $S_{\langle i\rangle}$ with $\mathbb{P}(S_{\langle i\rangle} > s) = \mathrm{e}^{-s}$, $s > 0$. Equip the sequence of types $\{\xi_{\langle i\rangle}\}_{i=0}^{\infty}$ with a random walk structure with mutually independent and identically distributed increments $\eta_{\langle i\rangle} = \xi_{\langle i+1\rangle} - \xi_{\langle i\rangle}$ satisfying

$$\mathbb{P}_\xi(\eta_{\langle i\rangle} = -\alpha_k) = \frac{\hat{c}(\alpha_k)}{\sum_{j=1}^{m_c}\hat{c}(\alpha_j)}, \quad k = 1, \ldots, m_c. \tag{15}$$

Then Eq. (14) can be written as

$$\hat{u}(t,\xi) = \mathbb{E}_\xi\left\{\mathbf{1}_{[S_{\langle 0\rangle} > t]} m(t,\xi)\hat{u}_0(\xi) + \mathbf{1}_{[S_{\langle 0\rangle} \leqslant t]} m(S_{\langle 0\rangle}, \xi)\hat{u}(t - S_{\langle 0\rangle}, \xi + \eta_{\langle 0\rangle})\right\} \tag{16}$$

hinting that $\hat{u}$ can be represented as the expected value of a random product. Define termination times by

$$T_{\langle 0\rangle} = t, \quad T_{\langle i+1\rangle} = t - (S_{\langle 0\rangle} + \cdots + S_{\langle i\rangle}), \quad i = 0, 1, \ldots$$

and let $N = N(\tau(t,\xi)) = \inf\{i : S_{\langle i\rangle} > T_{\langle i\rangle}\}$. The random variable N gives the total number of vertices of τ and has a Poisson distribution with mean t, namely $\mathbb{P}(N = n) = \frac{1}{n!}\mathrm{e}^{-t}t^n$. Define the multiplicative functional $\mathsf{X}(t,\xi) = \mathsf{X}(\tau(t,\xi))$ of the random collections $\{\xi_{\langle i\rangle}\}_{i=0}^{N}$, $\{S_{\langle i\rangle}\}_{i=0}^{N}$ by:

$$\mathsf{X}(\tau(t,\xi)) = \left(\prod_{i=0}^{N-1} m'(S_{\langle i\rangle}, \xi_{\langle i\rangle})\right) m(T_{\langle N\rangle}, \xi_{\langle N\rangle})\hat{u}_0(\xi_{\langle N\rangle}), \tag{17}$$

where for $N = 0$ the product in parenthesis is taken to be one. The following holds

Theorem 2.1. *For $t > 0$, $\xi = \xi_{\langle 0\rangle}$,*

$$\hat{u}(t,\xi) = \mathbb{E}_\xi\, \mathsf{X}(\tau(t,\xi))$$

is a solution to the integral equation (14).

Proof. The finiteness of the expected value has to be established first. Let $\sigma \geqslant \sum_{i=1}^{m_c}\hat{c}(\alpha_k)$, $U \geqslant \max\{\hat{u}(b_j), j = 1, \ldots, m_u\}$, and note that for each ξ, $m(t,\xi) \leqslant \mathrm{e}^t$ and $m'(t,\xi) \leqslant \sigma\mathrm{e}^t$. Then

$$\mathbb{E}_\xi \mathsf{X}(\tau(t,\xi)) \leqslant U\mathrm{e}^t\mathbb{E}_\xi\left\{\prod_{i=0}^{N-1}\sigma\mathrm{e}^{S_{\langle i\rangle}}\right\} = U\mathrm{e}^t\sum_{n=0}^{\infty}\frac{t^n\mathrm{e}^{-t}}{n!}\sigma^n\mathrm{e}^{S_{\langle 0\rangle}+\cdots+S_{\langle n-1\rangle}} \leqslant U\mathrm{e}^t\sum_{n=0}^{\infty}\frac{t^n\mathrm{e}^{-t}}{n!}\sigma^n\mathrm{e}^t = U\mathrm{e}^{\sigma(t+1)}.$$

Let $t > 0$, $\xi = \xi_{\langle 0\rangle}$. By the lack of memory of the exponential distribution, (17) can be written recursively as

$$\mathsf{X}(\tau(t,\xi)) = \begin{cases} m(t,\xi_{\langle 0\rangle})\hat{u}_0(\xi_{\langle 0\rangle}) & \text{if } S_{\langle 0\rangle} > t, \\ m'(S_{\langle 0\rangle}, \xi_{\langle 0\rangle})\mathsf{X}(\tau(t - S_{\langle 0\rangle}, \xi_{\langle 1\rangle})) & \text{if } S_{\langle 0\rangle} \leqslant t. \end{cases} \tag{18}$$

Use the mutual independence of $S_{\langle i\rangle}$ and $\eta_{\langle i\rangle}$ to get

$$\begin{aligned}
\mathbb{E}_\xi\mathsf{X}(t,\xi) &= m(t,\xi)\hat{u}_0(\xi)\mathbb{P}(S_{\langle 0\rangle} > t) + \mathbb{E}_\xi\left\{m'(S_{\langle 0\rangle},\xi)\mathsf{X}(t - S_{\langle 0\rangle}, \xi + \eta_{\langle 0\rangle})\mathbf{1}_{[S_{\langle 0\rangle}\leqslant t]}\right\} \\
&= m(t,\xi)\mathrm{e}^{-t}\hat{u}_0(\xi) + \int_0^t \mathrm{e}^{-s}m'(s,\xi)\mathbb{E}_\xi\left\{\mathsf{X}(t-s, \xi + \eta_{\langle 0\rangle})|S_{\langle 0\rangle} = s\right\}\mathrm{d}s \\
&= m(t,\xi)\mathrm{e}^{-t}\hat{u}_0(\xi) + \int_0^t \mathrm{e}^{-s}m'(s,\xi)\mathbb{E}_\xi\left\{\mathsf{X}(t-s, \xi + \eta_{\langle 0\rangle})|S_{\langle 0\rangle} = s\right\}\mathrm{d}s \\
&= m(t,\xi)\mathrm{e}^{-t}\hat{u}_0(\xi) + \int_0^t \mathrm{e}^{-s}m'(s,\xi)\sum_{i=1}^{m_c}\left\{\frac{\hat{c}(\alpha_k)}{\sum_{i=1}^{m_c}\hat{c}(\alpha_k)}\mathbb{E}_\xi\mathsf{X}(t-s,\xi-\alpha_i)\right\}\mathrm{d}s. \qquad \square
\end{aligned}$$

Heuristically, the underlying stochastic process can be thought of as a "construction" of $\tau(t,\xi)$ as follows: fix $\xi_{\langle 0\rangle}=\xi$, generate $S_{\langle 0\rangle}$ and compare its value to the termination time $T_{\langle 0\rangle}=t$. In the event $[S_{\langle 0\rangle}>t]$ the tree has no further vertices. If $[S_{\langle 0\rangle}\leqslant t]$, the vertex $\langle 1\rangle$is created and it is assigned a random type $\xi_{\langle 1\rangle}=\xi_{\langle 0\rangle}+\eta_{\langle 0\rangle}$ (see Fig. 1). Repeat this process at $\langle 1\rangle,\langle 2\rangle,\ldots$ until no more new vertices arise. Assign the correct multipliers to the N resulting nodes, and evaluate (17).

2.1. Modeling

As a first step, $\hat{u}(t,\xi)$ is approximated by a Monte Carlo estimation of $\mathbb{E}\mathsf{X}(\tau(t,\xi))$ given by Eq. (17). Let ξ and t be fixed. The construction of $\tau(t,\xi)$ can be modified in such a way that only realizations that contribute to the mean of $\mathsf{X}(t,\xi)$ are preformed, i.e., realizations with $\hat{u}_0(\xi_{\langle N\rangle})\neq 0$. For this, condition the expected value of (17) on $\xi_{\langle 0\rangle}=\xi$, $N=n$ and $\xi_{\langle n\rangle}=\beta_j$, to get

$$\hat{u}(t,\xi)=\sum_{j=1}^{m_u}\sum_{n=0}^{\infty}\mathbb{E}_{\xi}\left\{\left[\prod_{i=0}^{n-1}m'(S_{\langle i\rangle},\xi_{\langle i\rangle})\right]m(T_{\langle n\rangle},\beta_j)\hat{u}_0(\beta_j)\right\}\frac{\mathrm{e}^{-t}t^n}{n!}\mathbb{P}_{\xi}(\xi_{\langle n\rangle}=\beta_j). \tag{19}$$

The Monte Carlo simulation of the expected value in Eq. (19) can be done as follows. For each t, n, and β_j, perform an appropriate number of random "backward walks" $\{\xi_{\langle i\rangle}\}_{i=n}^{0}$ with $\xi_{\langle n\rangle}=\beta_j$, $\xi_{\langle i\rangle}=\beta_j-\eta_{\langle n-1\rangle}-\cdots-\eta_{\langle i\rangle}$, $i=n-1,\ldots,0$, and generate exponential times $\{S_{\langle i\rangle}\}_{i=0}^{n-1}$ conditioned to $[S_{\langle 0\rangle}+\cdots+S_{\langle n-1\rangle}<t]$. Then calculate the average of the product

$$\left[\prod_{i=0}^{n-1}m'(S_{\langle i\rangle},\xi_{\langle i\rangle})\right]m(t-(S_{\langle 0\rangle}+\cdots+S_{\langle n\rangle}),\beta_j)\hat{u}_0(\beta_j)$$

over all walks with $\xi_{\langle 0\rangle}=\xi$.

The probabilities $\mathbb{P}_{\xi}(\xi_{\langle n\rangle}=\beta_j)$, $j=1,\ldots,m_u$, can be computed from the nth power of the transition probability matrix of the Markov process $\{\xi_{\langle i\rangle}=\xi+\eta_{\langle 0\rangle}+\cdots+\eta_{\langle i-1\rangle}\}_{i\geqslant 1}$. The entries of this matrix are obtained from (15).

Some remarks are in order. First, recall that the conditional distribution of $(S_{\langle 0\rangle},S_{\langle 0\rangle}+S_{\langle 1\rangle},\cdots,S_{\langle 0\rangle}+\cdots+S_{\langle n\rangle})$ given $[S_{\langle 0\rangle}+\cdots+S_{\langle n\rangle}<t]$ is the same as the distribution of n increasingly ordered independent random variables each having the uniform distribution on $(0,t]$, (see [12] p. 280). Secondly, although the

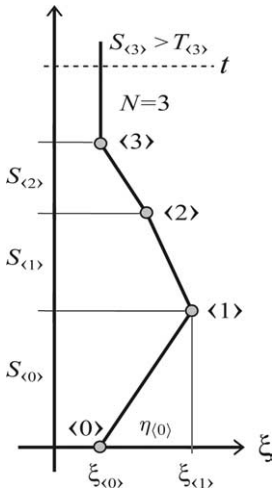

Fig. 1. Diagram of a realization of a tree τ for the diffusion equation with $N=3$. The value of the multiplicative functional is $\mathsf{X}(t,\xi)=m(S_{\langle 0\rangle},\xi_{\langle 0\rangle})m(S_{\langle 1\rangle},\xi_{\langle 1\rangle})m(S_{\langle 2\rangle},\xi_{\langle 2\rangle})m'(T_{\langle 3\rangle},\xi_{\langle 3\rangle})$.

summation over n in (19) is over all positive integers, the probability $\frac{e^{-t}t^n}{n!}\mathbb{P}_\xi(\xi_{\langle n\rangle}=\beta_j)$ decreases to zero very fast for moderate values of t, so only small trees have to be considered. For example, for values of t close to 1, trees with $N \geqslant 15$ have probability of the order 10^{-8}, and so their contribution to the mean can be neglected.

A Picard iteration of the integral equation (12) can be used to assess the accuracy of the Monte Carlo simulation and improve the results. Let $\hat{u}^{(0)}$ be the approximation of $\mathbb{E}_k\mathsf{X}(\tau(t,\xi))$, and for $n \geqslant 0$, define

$$\hat{u}^{(n+1)}(t,\xi) = e^{-\frac{a^2}{2}\xi^2 t}\hat{u}_0^{(n)}(\xi) + \int_0^t e^{-\frac{a^2}{2}\xi^2 s}\left\{\sum_{i=1}^{m_c}\hat{c}(\alpha_i)\,\hat{u}^{(n)}(t-s,\xi-\alpha_i)\right\}ds, \tag{20}$$

a sequence in the space of almost everywhere bounded functions $L^\infty(\mathbb{R})$. A solution to the diffusion equation in Fourier space is a fixed point of Eq. (20), and the error of the nth approximation to the solution can be measured by

$$E_n = \frac{\|\hat{u}^{(n+1)} - \hat{u}^{(n)}\|_\infty}{\|\hat{u}^{(n)}\|_\infty}.$$

Crucial to the algorithm presented here is the introduction of the multipliers in (13). Previous attempts using random times $S_{\langle i\rangle}$ with frequency-dependent mean and time-independent multipliers $m(\xi)$, led to very unstable and non-convergent simulations (S. Dobson, E. Thomann, personal communication). This problem is corrected by the definition of $m(\xi, t)$ used here.

2.2. Example

Consider Eq. (10) with data $c(x) = \cos x$, $u_0(x) = \sin x$. The problem in Fourier space is

$$\hat{u}_t(t,\xi) = -\frac{a^2}{2}\xi^2\hat{u}(t,\xi) + \frac{1}{2}(\delta_1+\delta_{-1}) * \hat{u}(t,\xi), \quad \hat{u}_0(\xi) = \frac{1}{2}(\delta_1-\delta_{-1}). \tag{21}$$

Only trees (walks) with $\xi_{\langle N\rangle} = \pm 1$ have to be constructed, and the solution is expected to be an odd function. The number of generated walks of length N for each finishing should be large enough so all accessible values of $\xi_{\langle 0\rangle}$ are well sampled, and the values of the aggregates $S_{\langle 0\rangle}, S_{\langle 0\rangle}+S_1, \ldots, S_{\langle 0\rangle}+\cdots+S_{N-1}$ exhibit a good approximation to a uniform distribution on $(0,t]$. A simple heuristic rule is, for each N and t, to generate a number of trees proportional to Nt. The error for this particular example showed little change for values of the constant of proportionality over 1000, so this value was used. The iterates of (20) were computed for 10 equally spaced time points in $[0,1]$, and the integration in time was performed with a simple trapezoidal rule. A comparison between the consecutive iterates $n = 0,1$ and $n = 4,5$ is shown in Figs. 2 and 3. The observed errors listed below indicate a linear rate of convergence.

n	0	1	2	3	4	5
E_n	6.7×10^{-3}	5.9×10^{-4}	1.6×10^{-4}	7.0×10^{-5}	1.6×10^{-5}	3.2×10^{-6}

3. Viscous Burgers equation

Consider the viscous Burgers equation in $\mathbb{T} = [0,1)$ with periodic initial condition $u_0 = u_0(x)$, periodic forcing term $f = f(t,x)$, periodic boundary conditions, and viscosity $\nu > 0$,

$$\begin{aligned} &u_t(t) + \sqrt{2\pi}u(t)u_x(t) = \nu u_{xx}(t) + f, \quad t > 0,\\ &u(0^+,x) = u_0(x), \quad u(t,0) = u(t,1). \end{aligned} \tag{22}$$

A solution to Eq. (22) will remain periodic for $t > 0$, and therefore one has Fourier series representations for u_0, f, and u, with coefficients $\hat{u}_0(k,t)$, $\hat{f}(k,t)$ and $\hat{u}(k,t)$, respectively, $k \in \mathbb{Z}$.

Write $uu_x = \frac{1}{2}(u^2)_x$ and take Fourier transform of both sides of (22). Use the integrator factor $e^{\nu k^2 s}$, and multiply and divide by νk^2 inside the resulting integral, then

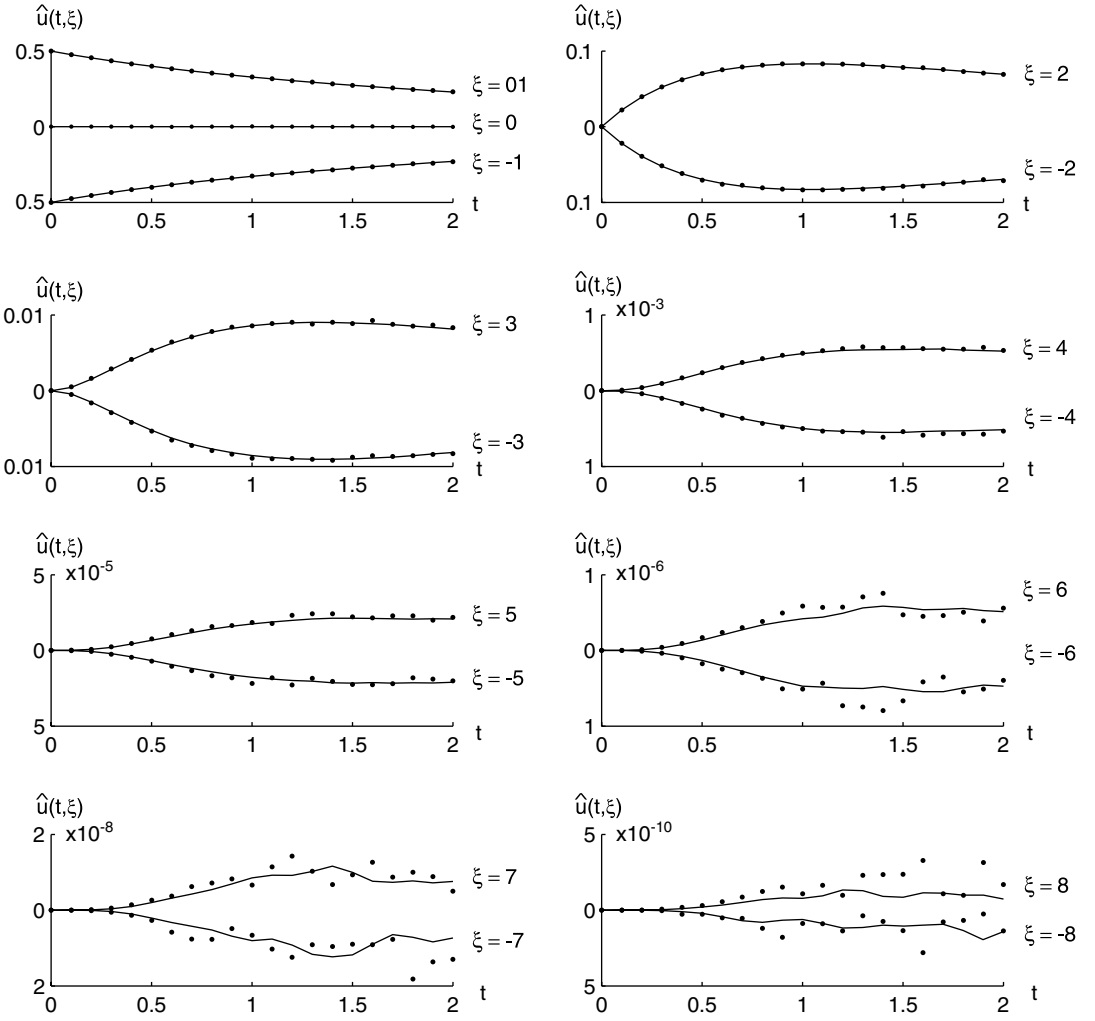

Fig. 2. Comparison between $\hat{u}^{(0)} = \mathbb{E}_\xi X$ (dots) and $\hat{u}^{(1)}$ (lines) for the diffusion equation with data given by (21).

$$\hat{u}(t,k) = \hat{u}_0(k)\mathrm{e}^{-\nu k^2 t} + \int_0^t \nu k^2 \mathrm{e}^{-\nu k^2 s}\left[\frac{1}{2}\frac{1}{\mathrm{i}\nu}\frac{\hat{u} * \hat{u}(t-s,k)}{k} + \frac{1}{2}\frac{2\hat{f}(t-s,k)}{\nu k^2}\right]\mathrm{d}s. \tag{23}$$

Now, make the following change of variables

$$\hat{w}(t,k) = \frac{1}{\mathrm{i}\nu}\hat{u}(t,k), \quad \hat{w}_0(k) = \hat{w}(0,k), \quad \hat{g}(t,k) = \frac{2\hat{f}(t,k)}{\mathrm{i}\nu^2 k^2}. \tag{24}$$

Then $\hat{w}$ satisfies the following equation

$$\hat{w}(t,k) = \hat{w}_0(k)\mathrm{e}^{-\nu k^2 t} + \int_0^t \nu k^2 \mathrm{e}^{-\nu k^2 s}\left[\frac{1}{2}\frac{\hat{w} * \hat{w}(t-s,k)}{k} + \frac{1}{2}\hat{g}(t-s,k)\right]\mathrm{d}s, \tag{25}$$

where the convolution $\hat{w} * \hat{w}$ is to be understood over $\mathbb{Z}$ as

$$\hat{w} * \hat{w}(t-s,k) = \sum_{j\in\mathbb{Z}} \hat{w}(t-s,j)\hat{w}(t-s,k-j). \tag{26}$$

The form of (25) is similar to that of (12), and the initial structure of the appropriate stochastic process can be guessed. Let $S_{\langle 0\rangle}$ be an exponentially distributed random variable of parameter νk^2, and let $c_{\langle 0\rangle}$ be a fair coin tossing with values in $\{0,1\}$. Eq. (25) can be written as

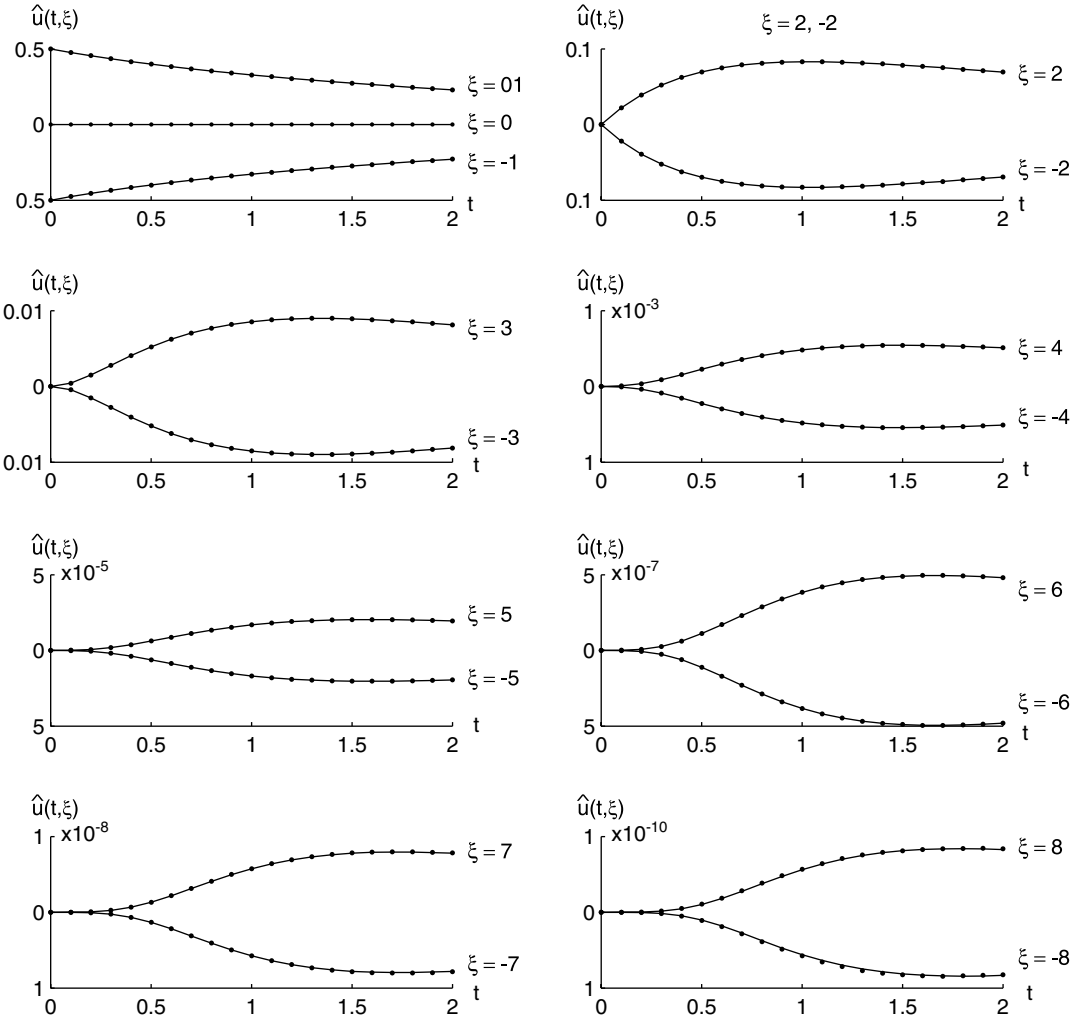

Fig. 3. Comparison between $\hat{u}^{(4)}$ (dots) and $\hat{u}^{(5)}$ (lines) for the diffusion equation with data given by (21).

$$\hat{w}(t,k) = \mathbb{E}\left\{ \mathbf{1}_{[S_{\langle 0\rangle} > t]} \hat{w}_0(k) + \mathbf{1}_{[S_{\langle 0\rangle} \leqslant t]} \left[c_{\langle 0\rangle} \frac{\hat{w} * \hat{w}(t - S_{\langle 0\rangle}, k)}{k} + (1 - c_{\langle 0\rangle}) \hat{g}(t - S_{\langle 0\rangle}, k) \right] \right\}. \tag{27}$$

Note that if a discrete probability density function on $j \in \mathbb{Z}$ is introduced in the summation (26), the convolution could be interpreted (for each k) as an average of products of $\hat{w}$ evaluated at random frequencies. The problem of finding such a density appears whenever stochastic cascades are used to solve partial differential equations in Fourier space, and is linked to the more general problem of establishing existence and regularity of solutions. In the more general case of the Navier–Stokes equations, the characterization of admissible densities for the frequencies of offspring vertices was solved with the introduction of "majorizing kernels" in [10,13,14].

It follows from Eq. (26) that any function decreasing as $o(j^{-\frac{1}{2}})$ can be used to give a full probabilistic representation of (25). This will provide the existence of solutions without restrictions on the support of $\hat{u}_0$ and $\hat{f}$. Some attempts to numerically model this problem have been made by S. Dobson, E. Thomann, A. Chorin A. and P. Stinis (personal communications).

Here the simplest possible probabilistic representation of the convolution (26) is used through the following assumption:

$$\text{there exists } K > 0 \text{ such that } \hat{f}(t,k) = \hat{u}_0(k) = 0 \quad \text{for all } k < K. \tag{28}$$

It is due to a theorem of F. and M. Riesz (see [15, p. 335]) that property (28) implies that f and u_0 belong to the Hardy space H^1, namely, the Banach space of functions with holomorphic extension to the unit disc with the norm

$$\|f\|_{H^p} = \lim_{r\to 1}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(r\mathrm{e}^{\mathrm{i}\theta}|\,\mathrm{d}\theta\right\}^{\frac{1}{p}}.$$

Assume (28) holds and let $J_{\langle 0\rangle}$ be a uniformly distributed random variable taking values in $\{K,\ldots,k-K\}$, then

$$\hat{w} * \hat{w}(t,k) = \begin{cases} 0, & K \leqslant k < 2K, \\ (k-2K+1)\ \mathbb{E}\{\hat{w}(t,J_{\langle 0\rangle})\hat{w}(t,k-J_{\langle 0\rangle})\}, & k \geqslant 2K. \end{cases}$$

Denote $p_{\langle 0\rangle} = (k-2K+1)^{-1}$, and write Eq. (27) as

$$\begin{aligned}\hat{w}(t,k) = {} & \mathbb{E}\left\{\mathbf{1}_{[S_{\langle 0\rangle}>t]}\hat{w}_0(k) + \mathbf{1}_{[S_{\langle 0\rangle}\leqslant t]}\left[c_{\langle 0\rangle}\frac{1}{p_{\langle 0\rangle}k}\hat{w}(t-S_{\langle 0\rangle},J_{\langle 0\rangle})\hat{w}(t-S_{\langle 0\rangle},k-J_{\langle 0\rangle}) + (1-c_{\langle 0\rangle})\hat{g}(t-S_{\langle 0\rangle},k)\right]\right\}\mathbf{1}_{[k\geqslant 2K]} \\ & + \mathbb{E}\left\{\mathbf{1}_{[S_{\langle 0\rangle}>t]}\hat{w}_0(k) + \mathbf{1}_{[S_{\langle 0\rangle}\leqslant t]}\left[\frac{1}{2p_{\langle 0\rangle}k}\hat{g}(t-S_{\langle 0\rangle},k)\right]\right\}\mathbf{1}_{[k<2K]}.\end{aligned} \tag{29}$$

From the construction of the appropriate stochastic process associated to (27), it will follow that $\hat{w}(t,k)$ satisfies (28) for all $t>0$.

For $k>0$ and $t>0$ consider a binary tree $\tau=\tau(t,k)$ with a particle of type k assigned to its root vertex $\langle 0\rangle$. Let $(\Omega,\mathscr{F},\mathbb{P}_k)$ be a probability space. Define a vertex-indexed stochastic process $\{k_{\langle v\rangle}\}_{\langle v\rangle\in\tau}$ with $k_{\langle 0\rangle}=k$, satisfying the conservation rule

$$k_{\langle v\rangle} = k_{\langle v1\rangle} + k_{\langle v2\rangle}, \quad \langle v\rangle \in \tau, \tag{30}$$

and with increments $k_{\langle v1\rangle} - k_{\langle v\rangle} = J_{\langle v\rangle}$ conditionally distributed as,

$$\mathbb{P}_k(J_{\langle v\rangle} = j | k_{\langle v\rangle}, c_{\langle v\rangle}) = \frac{1}{k_{\langle v\rangle}-2K+1}\mathbf{1}_{[k_{\langle v\rangle}\geqslant 2K]}\mathbf{1}_{[c_{\langle v\rangle}=1]} := p_{\langle v\rangle}. \tag{31}$$

Let $c_{\langle v\rangle}$ be fair coin tosses taking values on $\{0,1\}$. Introduce "waiting times" $S_{\langle v\rangle}$ with conditional exponential distributions given by

$$\mathbb{P}_k(S_{\langle v\rangle} > s \mid k_{\langle v\rangle}) = \mathrm{e}^{-\nu k^2_{\langle v\rangle}s}, \quad s>0.$$

Finally, for $\langle v\rangle\in\tau$, define termination times as

$$T_{\langle v\rangle} = t - \sum_{j=0}^{|\langle v\rangle|-1} S_{\langle v\rangle|j}, \quad T_{\langle 0\rangle} = t, \tag{32}$$

where $|\langle v\rangle|$ and $\langle v\rangle|j$ are defined in Section 1.

A multiplicative functional of $\{k_{\langle v\rangle}, S_{\langle v\rangle}, c_{\langle v\rangle}\}_{\langle v\rangle\in\tau}$ related to $\hat{w}$ can now be constructed in a similar way as in Section 2. Consider

$$\mathsf{X}(\tau(t,k)) = \mathsf{X}(t,k) = \prod_{\langle v\rangle\in\tau} M_{\langle v\rangle} \tag{33}$$

with multipliers given by

$$M_{\langle v\rangle} = \begin{cases} \hat{w}_0(k_{\langle v\rangle}) & \text{if } S_{\langle v\rangle} > T_{\langle v\rangle}, \\ \frac{1}{2}\hat{g}(T_{\langle v\rangle},k_{\langle v\rangle}) & \text{if } k_{\langle v\rangle} < 2K, S_{\langle v\rangle}\leqslant T_{\langle v\rangle}, \\ \hat{g}(T_{\langle v\rangle},k_{\langle v\rangle}) & \text{if } k_{\langle v\rangle}\geqslant 2K, S_{\langle v\rangle}\leqslant T_{\langle v\rangle}, c_{\langle v\rangle}=0, \\ \frac{1}{k_{\langle v\rangle}p_{\langle v\rangle}} & \text{if } k_{\langle v\rangle}\geqslant 2K, S_{\langle v\rangle}\leqslant T_{\langle v\rangle}, c_{\langle v\rangle}=1. \end{cases} \tag{34}$$

Then the following holds:

Theorem 3.1. *Assume* $\hat{f}(t,k) = \hat{u}_0(k) = 0$ *for all* $k < K$, *and that there is* $\alpha \geqslant 0$ *such that* $|\hat{u}_0(k)| \leqslant v\mathrm{e}^{-\alpha k}$, $|\hat{f}(t,k)| \leqslant \frac{v^2k^2}{2}\mathrm{e}^{-\alpha k}$ *for* $k \geqslant K, t > 0$. *Then Burgers equation (22) has a unique solution* $\hat{u}$. *Moreover*,

$$\hat{u}(t,k) = 0, \quad k < K \quad \text{and} \quad |\hat{u}(t,k)| \leqslant v\mathrm{e}^{-\alpha k}, \quad k \geqslant K,\ t > 0,$$

and $\hat{u}(t,k)$ *is explicitly given by*

$$\hat{u}(t,k) = iv\ \mathbb{E}_k\mathsf{X}(\tau(t,k)).$$

Proof. The bounds on $\hat{u}$ and $\hat{f}$ combined with (24), and the definition of $p_{\langle v\rangle}$ give the following bound for the random product in (34),

$$\mathsf{X}(t,k) \leqslant \exp\left\{-\alpha \sum_{\langle v\rangle\in\tau\setminus\tau_{\mathrm{B}}} k_{\langle v\rangle}\right\} \prod_{\langle v\rangle\in\tau_{\mathrm{B}}} \frac{k_{\langle v\rangle} - 2K + 1}{k_{\langle v\rangle}},$$

where τ_{B} is the set of vertices where branching occurs, namely,

$$\tau_{\mathrm{B}} = \{\langle v\rangle \in \tau : k_{\langle v\rangle} \geqslant 2K \text{ or } S_{\langle v\rangle} \leqslant T_{\langle v\rangle} \text{ or } c_{\langle v\rangle} = 1\}.$$

Since $\alpha \geqslant 0$ and $0 < K \leqslant k_{\langle v\rangle}$, then $|\mathbb{E}_k\mathsf{X}(\tau(t,k))| < \infty$. Moreover, the conservation rule (30) gives $\sum_{\langle v\rangle\in L} k_{\langle v\rangle} = k$, so the estimate for $|\hat{u}(t,k)|$ holds and $\hat{u}(t,k) = 0$ for $k < K$. Due to the Markovian character of the waiting times $S_{\langle v\rangle}$, and the mutual independence between $c_{\langle v\rangle}$ and $J_{\langle v\rangle}$, the following recursive representation of X is available,

$$\mathsf{X}(\tau(t,k_{\langle 0\rangle})) = \begin{cases} \hat{w}_0(k_{\langle 0\rangle}) & \text{if } S_{\langle 0\rangle} > T_{\langle 0\rangle}, \\ \frac{1}{2}\hat{g}(T_{\langle 0\rangle}, k_{\langle 0\rangle}) & \text{if } k_{\langle 0\rangle} < 2k^0, S_{\langle 0\rangle} \leqslant T_{\langle 0\rangle}, \\ \hat{g}(T_{\langle 0\rangle}, k_{\langle 0\rangle}) & \text{if } k_{\langle 0\rangle} \geqslant 2K, S_{\langle 0\rangle} \leqslant T_{\langle 0\rangle}, c_{\langle 0\rangle} = 0, \\ \frac{1}{k_{\langle 0\rangle}p_{\langle 0\rangle}}\mathsf{X}(\tau(T_{\langle 0\rangle}, k_{\langle 1\rangle}))\mathsf{X}(\tau(T_{\langle 0\rangle}, k_{\langle 2\rangle})) & \text{if } k_{\langle 0\rangle} \geqslant 2K, S_{\langle 0\rangle} \leqslant T_{\langle 0\rangle}, c_{\langle 0\rangle} = 1, \end{cases} \tag{35}$$

Conditioning on the cases of (35) gives that $\mathbb{E}_k\mathsf{X}(\tau(t,k))$ satisfies Eq. (23). □

3.1. Modeling

Because of the binary tree structure of τ, conditioning on the frequencies at the terminating vertices does not simplify the computation of $\mathsf{X}(\tau)$ as it did for the example in Section 2. Here, the realizations of the multiplicative functional can be done constructing trees from the root vertex following Eq. (35) (see Fig. 4). The root particle of type $k_{\langle 0\rangle} = k$ holds for the exponential time $S_{\langle 0\rangle}$ which is compared to $T_{\langle 0\rangle} = t$. If $[S_{\langle 0\rangle} > t]$, then no further vertices are used. If $[S_{\langle 0\rangle} < t]$ occurs, a coin $c_{\langle 0\rangle}$ is tossed. In the event $c_{\langle 0\rangle} = 0$, again the construction stops. If $c_{\langle 0\rangle} = 1$, branching occurs, and the new vertices $\langle 1\rangle$ and $\langle 2\rangle$ are created with random types $k_{\langle 1\rangle} = J_{\langle 0\rangle}$ and $k_{\langle 2\rangle} = k_{\langle 0\rangle} - J_{\langle 0\rangle}$, respectively. The same process is followed independently with trees rooted in $\langle 1\rangle$ and $\langle 2\rangle$. Whenever a vertex $\langle v\rangle$ has type $k_{\langle v\rangle} < 2K$, then $S_{\langle v\rangle}$ is compared to the respective termination time but no coin is tossed, and no branching occurs. The multipliers are then assigned to the vertices according to (34).

The integral equation (23) can be used to test the error in any numerical estimation of the expected value in Theorem 3.1. Define $\hat{u}^{(0)}$ to be the approximation provided by Monte Carlo algorithm, and define the following iterates:

$$\hat{u}^{(n+1)}(t,k) = \hat{u}_0(k)\mathrm{e}^{-vk^2t} + \int_0^t vk^2\mathrm{e}^{-vk^2s}\left[\frac{\hat{u}^{(n)} * \hat{u}^{(n)}(t-s,k)}{2\mathrm{i}vk} + \frac{\hat{f}(t-s,k)}{vk^2}\right]\mathrm{d}s. \tag{36}$$

The sequence $\{\hat{u}_{(n)}\}_{n\geqslant 0}$ forms a Picard iteration of which the solution $\hat{u}$ is a fixed point. The error at each term can be measured with

$$E_n = \frac{\|\hat{u}^{(n+1)} - \hat{u}^{(n)}\|_\infty}{\|\hat{u}^{(n)}\|_\infty}.$$

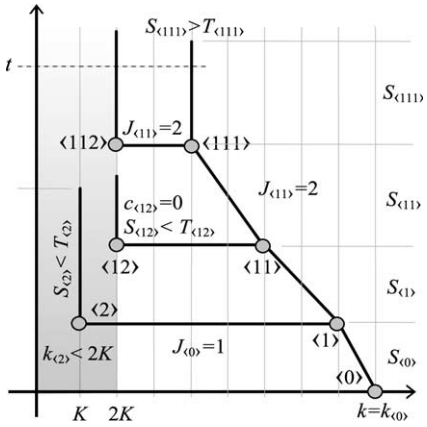

Fig. 4. Diagram of a realization of a tree τ for Burgers equation with $N = 3$. The value of the multiplicative functional is $\mathsf{X}(t,k) = \frac{1}{k_{\langle 0\rangle}p_{\langle 0\rangle}}\frac{1}{k_{\langle 1\rangle}p_{\langle 1\rangle}}\frac{\hat{g}(T_{\langle 2\rangle},k_{\langle 2\rangle})}{2}\frac{1}{k_{\langle 11\rangle}p_{\langle 11\rangle}}\hat{g}(T_{\langle 12\rangle},k_{\langle 12\rangle})\frac{1}{k_{\langle 111\rangle}p_{\langle 111\rangle}}\hat{w}_0(k_{\langle 112\rangle})\hat{w}_0(k_{\langle 111\rangle})$.

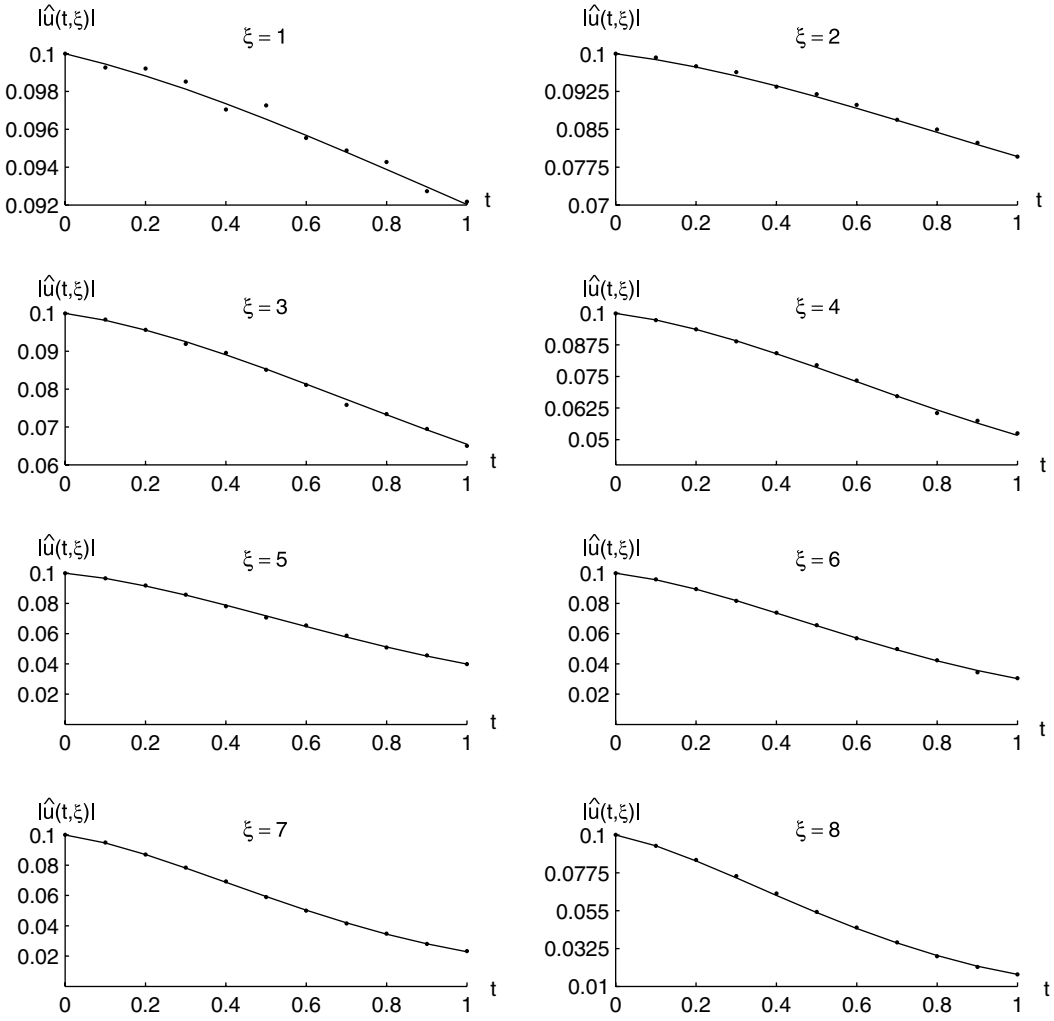

Fig. 5. Comparison between $\hat{u}^{(0)} = \mathbb{E}_\xi \mathsf{X}$ (dots) and $\hat{u}^{(1)}$ (lines) for Burgers equation with data given by (37).

3.2. Example

Consider the Fourier transformed Burgers equation with data given by

$$\hat{u}_0(k) = \mathrm{i}v, \quad \hat{f}(t,k) = \frac{\mathrm{i}v^2k^2}{2}(1-t)^2, \quad t \in (0,1), \ \ k \geqslant 1. \tag{37}$$

By Theorem 3.1, $\hat{u}(t,k) = \mathrm{i}v\mathbb{E}_k\,\mathsf{X}(\tau(t,k))$ is the unique solution to this equation. A numerical approximation to $\hat{u}$ was calculated for $k = 1,\ldots,8$ and 10 discrete time points in $[0,1]$. The number of computed realizations of $\mathsf{X}(\tau)$ was proportional to t for frequencies $k_{\langle 0\rangle} < 2K$, and proportional to $(t+k-2K)$ for frequencies $k_{\langle 0\rangle} \geqslant 2K$. The proportionality constant used was 3000. Some terms of the sequence $\{\hat{u}^{(n)}\}$ defined by (36) were calculated using a simple trapezoidal rule for the integration on time. A comparison between the complex norm of the consecutive iterates for $n = 0,1$ and $n = 4,5$ is shown in Figs. 5 and 6. The observed errors, as defined by (3.1), are listed below.

n	0	1	2	3	4	5
E_n	1.8×10^{-2}	2.1×10^{-3}	4.0×10^{-4}	1.0×10^{-4}	2.1×10^{-5}	2.2×10^{-6}

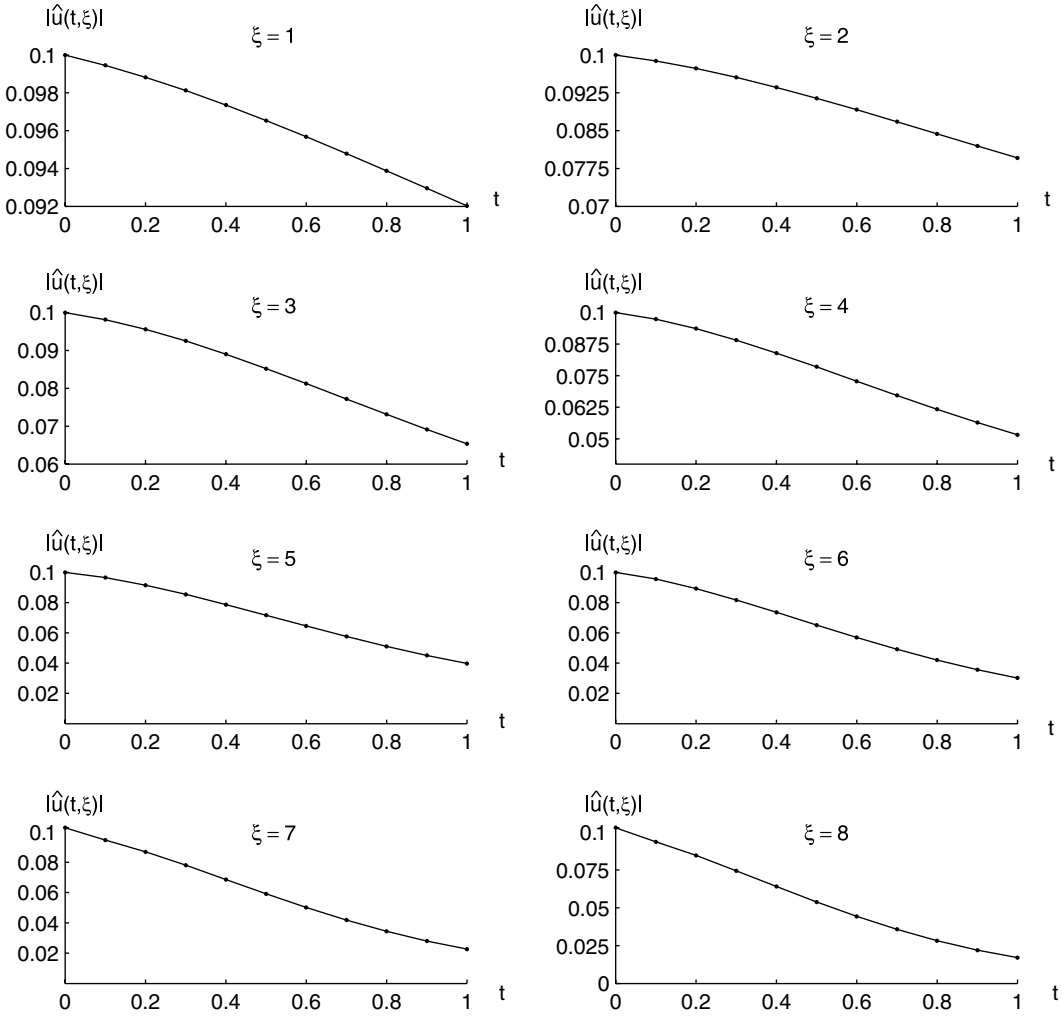

Fig. 6. Comparison between $\hat{u}^{(4)}$ (dots) and $\hat{u}^{(5)}$ (lines) for Burgers equation with data given by (37).

4. Concluding remarks

Monte Carlo techniques are developed to model the random multiplicative cascades associated with two partial differential equations: linear diffusion and viscous Burgers equation. The Fourier transformed solution of the PDE is written as the expected value of a random aggregate of its main components, i.e., initial condition, nonlinear terms and forcing terms. These aggregates are evaluated at random frequencies $\xi_{\langle v\rangle}$ (or $k_{\langle v\rangle}$), and random times $S_{\langle v\rangle}$, indexed by $\langle v\rangle\in\tau$, where τ has a tree structure. The choice on the distributions of $\xi_{\langle v\rangle}$ and $S_{\langle v\rangle}$ determine the limitations and scope of the multiplicative cascade representation.

Exponential waiting times $S_{\langle v\rangle}$ with parameter dependent on $k_{\langle v\rangle}$ arose naturally in both examples presented here (see Eqs. (12) and (23)). Computational stability considerations led to the removal of this dependence for the case of the diffusion equation. The exponential distribution has the advantage of giving Markov structure to the resulting stochastic process (see proof of Theorem 2.1), and it also simplifies the numerical modeling. Multiplicative cascade representations using distributions for the waiting times different than exponential are considered in [14].

In the examples presented here, the transition distributions for the frequency process $\{\xi_{\langle v\rangle}\}_{\langle v\rangle\in\tau}$ are chosen so both the analytical and the modeling problem are considerably simplified. This selection imposes restrictions on the PDE's data for which a multiplicative cascade representation gives a solution. There is however, no "physical" reason behind the choices made here. The identification of processes $\{\xi_{\langle v\rangle}\}_{\langle v\rangle\in\tau}$ that correspond with the physical situations the partial differential equations arise from is still unexplored. Simulations such as those presented here can be used to study the relationship between this processes in frequency space, and known qualitative features of the solutions.

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